

Running condensate in moving superfluid

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Abstract – A possibility of the condensation of excitations with non-zero momentum in moving superfluid media is considered in terms of the Ginzburg-Landau model. The results might be applicable to the superfluid ⁴He, ultracold atomic Bose gases, various superconducting and neutral systems with pairing, like ultracold atomic Fermi gases and the neutron component in compact stars. The order parameters, the energy gain, and critical velocities are found.

Introduction. – A possibility of the condensation of rotons in the He-II, moving in a capillary at zero temperature with a flow velocity exceeding the Landau critical velocity, was suggested in [1]. In [2] the condensation of excitations with non-zero momentum in various moving non-relativistic and relativistic media (not necessarily superfluid) was studied. A possibility of the formation of a condensate of zero-sound-like modes with a finite momentum in normal Fermi liquids at non-zero temperature was further discussed in [3]. The condensation of excitations in cold atomic Bose gases was recently studied in [4]. The works [1,2,4] disregarded a non-linear interaction between the “mother” condensate of the superfluid and the condensate of excitations. The condensation of excitations in superfluid systems at finite temperatures, *i.e.*, in the presence of a normal subsystem, also has not been studied yet.

The key idea is as follows [1,2]. When a medium moves as a whole with respect to a laboratory frame with a velocity higher than a certain critical velocity, it may become energetically favorable to transfer a part of its momentum from particles of the moving medium to a condensate of Bose excitations with a non-zero momentum $k \neq 0$. It would happen, if the spectrum of excitations is soft in some region of the momenta. Whether the system is moving linearly with a constant velocity or it is resting, is indistinguishable according to the Galilei invariance. Thus, there should still exist a physical mechanism allowing to produce excitations. The excitations can be created near a wall, which singles out the laboratory reference frame,

or they can be produced by interactions among particles of the normal subsystem at non-zero temperature. They can be also generated provided the motion is non-inertial, *e.g.*, in the case of a rotating system.

In the He-II there exists a branch of roton excitations [5,6]. The typical value of the energy of the excitations $\Delta_r = \epsilon(k_r)$ at the roton minimum for $k = k_r$ depends on the pressure and temperature. According to [7], for the saturated vapor pressure $\Delta_r = 8.71$ K at $T = 0.1$ K and $\Delta_r = 7.63$ K at $T = 2.10$ K, and $k_r \simeq 1.9 \cdot 10^8$ h/cm in the whole temperature interval. (We put the Boltzmann constant $k_B = 1$). An appropriate branch of excitations may exist also in normal Fermi liquids [3] and in cold Bose [4,8] and Fermi [9] atomic gases. In neutral Fermi liquids with the singlet pairing, characterized by the pairing gap Δ , there exist [10] the low-lying Anderson-Bogoliubov mode of excitations with $\epsilon(k) = kv_F/\sqrt{3}$ for $k \rightarrow 0$ and $\epsilon(k) \rightarrow 2\Delta$ for large k ($k \lesssim 2p_F$), and the Schmid mode $\epsilon(k) \simeq 2\Delta$. Here p_F is the Fermi momentum, v_F is the Fermi velocity. In charged Fermi liquids with the singlet pairing there is also the suitable low-lying Carlson-Goldman mode starting at zero energy for a small momentum and reaching the value $\epsilon = 2\Delta$ for large k .

Below, we study a possibility of the condensation of excitations in the state with a non-zero momentum in moving media in the presence of the superfluid subsystem. The systems of our interest are neutral bosonic superfluids, such as the superfluid ⁴He and cold Bose atomic gases, and systems with the Cooper pairing, like the neutron liquid in neutron star interiors, cold Fermi atomic gases or

charged superfluids, as paired protons in neutron star interiors and paired electrons in metallic superconductors. In contrast to previous works we take into account that the superfluid subsystem and the bosonic excitations should be described in terms of the very same macroscopic wavefunction. Also, our consideration is performed for non-zero temperature, *i.e.*, the presence of the normal component is taken into account.

Ginzburg-Landau functional. — We start with expression for the Ginzburg-Landau (GL) free-energy density of the superfluid subsystem in its rest reference frame for the temperature $T < T_c$, [5, 6]:

$$F_{\text{GL}}[\psi] = \frac{c_0}{2} |\hbar \nabla \psi|^2 - a(t) |\psi|^2 + \frac{1}{2} b(t) |\psi|^4, \\ a(t) = a_0 t^\alpha, \quad b(t) = b_0 t^\beta, \quad t = (T_c - T)/T_c. \quad (1)$$

Here T_c is the critical temperature of the second-order phase transition, and c_0 , a_0 and b_0 are phenomenological parameters. When treated within the mean-field approximation, the functional F_{GL} should be an analytic function of t . Then, from the Taylor expansion of F_{GL} in t it follows that $\alpha = 1$, $\beta = 0$. The width of the so-called fluctuation region, wherein the mean-field approximation is not applicable, is evaluated from the Ginzburg criterion: in this region of temperatures in the vicinity of T_c , long-range fluctuations of the order parameter are mostly probable, *i.e.* their probability is $W \sim e^{-F_{\text{GL}}^{\text{eq}} V_{\text{fl}}/T} \sim 1$, where $F_{\text{GL}}^{\text{eq}}$ is the equilibrium value, $V_{\text{fl}} \sim \xi^3$ is the minimal volume of the fluctuation of the order parameter, the coherence length ξ is the typical length scale characterizing the order parameter.

For metallic superconductors the fluctuation region proves to be very narrow and the mean-field approximation holds for almost any temperatures below T_c , except a tiny vicinity of T_c . Thus, neglecting this narrow fluctuation region one may use $\alpha = 1$, $\beta = 0$ in (1). For the fermionic systems with the singlet pairing, in the weak-coupling (BCS) approximation the parameters can be extracted from the microscopic theory [5]:

$$c_0 = 1/2m_{\text{F}}^*, \quad a_0 = 6\pi^2 T_c^2 / (7\zeta(3)\mu), \quad b_0 = a_0/n, \quad (2)$$

where m_{F}^* stands for the effective fermion mass ($m_{\text{F}}^* \simeq m_{\text{F}}$ in the weak-coupling limit), $n = p_{\text{F}}^3 / (3\pi^2 \hbar^3)$ is the particle number density, and the fermion chemical potential is $\mu \simeq \epsilon_{\text{F}} = p_{\text{F}}^2 / (2m_{\text{F}}^*)$. The function $\zeta(x)$ is the Riemann ζ -function and $\zeta(3) = 1.202$. The values of the parameters are obtained for $t \ll 1$. With these BCS parameters we have $|\psi|^2 = nt$ and the pairing gap $\Delta = T_c \sqrt{\frac{8\pi^2 t}{7\zeta(3)}}$, see [11].

For He-II the fluctuations prove to be important for all temperatures below T_c [6]. Including long-range fluctuations, the coefficients of the Ginzburg-Landau functional are now renormalized: due to a divergency (logarithmic or power-law-like) of the specific heat at the critical temperature T_c , quantities a and b in eq. (1) become non-analytic

functions of t with non-integer α and β . When the contributions of long-range fluctuations are completely taken into account the Ginzburg parameter $F_{\text{GL}}^{\text{eq}} \xi^3$ must become independent of the temperature. Since $F_{\text{GL}}^{\text{eq}} \propto t^{2\alpha-\beta}$ and $\xi \propto t^{-\alpha/2}$, we obtain $\alpha/2 - \beta = 0$. Using the experimental fact that the specific heat of the He-II contains no power divergence at $T \rightarrow T_c$, we get $2\alpha - \beta - 2 = 0$. From these two relations we find $\alpha = 4/3$, $\beta = 2/3$. Other parameters of He-II at the saturated vapor pressure are [6]: $T_c = 2.17 \text{ K}$, $a_0/T_c^{4/3} = 1.11 \cdot 10^{-16} \text{ erg/K}^{4/3}$, $b_0/T_c^{2/3} = 3.54 \cdot 10^{-39} \text{ erg} \cdot \text{cm}^3/\text{K}^{2/3}$. This parameterization holds for $10^{-6} < t < 0.1$, but for rough estimates can be used up to $t = 1$. *E.g.*, using eq. (1), with the helium atom mass $m = 6.6 \cdot 10^{-24} \text{ g}$ we evaluate the He-II mass-density as $ma_0/b_0 \simeq 0.3 \text{ g/cm}^3$, which is of the order of the experimental value $\rho_{\text{He}} = 0.15 \text{ g/cm}^3$ at $P = 0$.

Moving cold superfluid. — We consider now a superfluid moving with a constant velocity \vec{v} parallel to a wall. The latter singles out the laboratory frame and an interaction of the fluid with the wall may lead to creation of excitations in the fluid. We start with the case of vanishing temperatures. The whole medium is superfluid and the amplitude of the order parameter can be related to the particle density $\rho_s = mn = m|\psi_{\text{in}}|^2 = ma_0/b_0$, ψ_{in} is the order parameter in the absence of the excitations (“in”-state). The energy of the medium in the laboratory frame is $E_{\text{in}} = mnv^2/2 - b_0 n^2/2$.

When the speed of the flow, v , exceeds the Landau critical velocity, v_c^{L} , near the wall there may appear excitations with the momentum $k = k_0$ and the energy $\epsilon(k_0)$ as calculated in the rest frame of the superfluid, where the momentum k_0 corresponds to the minimum of the ratio $\epsilon(k)/k$. For instance, for He-II the spectrum $\epsilon(k)$ is the standard phonon+roton spectrum, normalized as $\epsilon(k) \propto k$ for small k . The appearance of a large number of excitations motivates us to assume that for $v > v_c^{\text{L}}$ in addition to the mother condensate the excitations may form a new condensate with the momentum $k = k_0 \neq 0$, (“fin”-state). The momentum k_0 should be found now from minimization of the free energy. As we have noticed before, in previous works [1, 2, 4] it was assumed that the condensate of excitations decouples from the mother condensate. Now we are going to take into account that the condensate of excitations is described by the very same macroscopic wave-function as the mother condensate. Then the resulting order parameter ψ_{fin} is the sum of the contributions of the mother condensate, ψ , and the condensate of excitations, ψ' , *i.e.*, $\psi_{\text{fin}} = \psi + \psi'$. We choose the simplest form of the order parameter for the condensate of excitations,

$$\psi' = \psi'_0 e^{-i(\epsilon(k_0)t - \vec{k}_0 \vec{r})/\hbar}, \quad (3)$$

with the amplitude ψ'_0 , being constant for the case of the homogeneous system that we consider. The particle number conservation yields

$$n = |\psi + \psi'|^2 = |\psi|^2 + |\psi'_0|^2 + \dots \quad (4)$$

Here ellipses stand for the spatially oscillating term, which vanishes after the averaging over the system volume.

As the particle density, the initial momentum density is redistributed in our case between the fluid and the condensate of excitations:

$$\rho_s \vec{v} = (\rho_s - m |\psi'_0|^2) \vec{v}_{\text{fin}} + (\vec{k}_0 + m \vec{v}_{\text{fin}}) |\psi'_0|^2. \quad (5)$$

The energy of the moving matter in the presence of the condensate of excitations is

$$E_{\text{fin}} = \frac{\rho_s v_{\text{fin}}^2}{2} + (\epsilon(k_0) + \epsilon_{\text{bind}}) |\psi'_0|^2 - a_0 |\psi|^2 + \frac{b_0 n^2}{2}, \quad (6)$$

where the energy of the excitation $\epsilon(k)$ should be counted from the binding energy of a particle in the condensate at rest $\epsilon_{\text{bind}} = \partial E_{\text{in}}(v=0)/\partial n = -b_0 n = -a_0$. Replacing eq. (5) in (6) and using eq. (4) we express the change of the volume-averaged energy density owing to the appearance of the condensate of excitations, $\delta \bar{E} = \bar{E}_{\text{fin}} - \bar{E}_{\text{in}}$, as

$$\delta \bar{E} = (\epsilon(k_0) - k_0 v) |\psi'_0|^2 + k_0^2 |\psi'_0|^4 / (2\rho_s). \quad (7)$$

Minimizing this functional with respect to ψ'_0 we obtain the condensate amplitude

$$|\psi'_0|^2 = (\rho_s / k_0) (v - v_c^L) \theta(v - v_c^L), \quad v_c^L = \epsilon(k_0) / k_0. \quad (8)$$

From (5) we find that because of condensation of excitations with $k \neq 0$ the flow is decelerated to the velocity $v_{\text{fin}} = v_c^L$. The volume-averaged energy density gain due to the appearance of the condensate of excitations is

$$\delta \bar{E} = -\frac{1}{2} \rho_s (v - v_c^L)^2 \theta(v - v_c^L). \quad (9)$$

Minimization of $\delta \bar{E}$ with respect to k_0 gives the condition $dv_c^L/dk_0 = 0$, which is exactly the condition for the standard Landau critical velocity. The condensate of excitations appears by a second-order phase transition. The amplitude of the condensate of excitations (8) grows with the velocity, whereas the amplitude of the mother condensate decreases. The value $|\psi|^2$ vanishes when $v = v_{c2}$, the second critical velocity, at which $|\psi'_0|^2 = n$ according to eq. (4). The value of the second critical velocity v_{c2} is evaluated from (8) as $v_{c2} = v_c^L + k_0/m$. When the mother condensate disappears, at $v = v_{c2}$, the excitation spectrum is cardinally reconstructed, and the superfluidity destruction occurs as a first-order phase transition. We assume that for $v > v_{c2}$ the excitation spectrum has no low-lying local minimum at finite momentum. Then the amplitude $|\psi'_0|^2$ jumps from n to 0 and $\delta \bar{E}$ jumps from $\delta \bar{E}(v_{c2}) = -\rho k_0^2 / (2m^2)$ to 0 at $v = v_{c2}$. The case, when in the absence of the mother condensate the spectrum of Bose excitations has a low-lying local minimum at $k \neq 0$, has been considered in [2, 3]. Note that in practice the reconstruction of the spectrum may occur for a smaller velocity than that we have estimated. For example, for fermionic superfluids it always should be $|\psi'_0|^2 \ll n$, otherwise the Fermi sea itself would be destroyed.

In moving superfluids there exist excitations of the type of vortex rings. The energy of the vortex is estimated as $\epsilon^{\text{vort}} = 2\pi^2 \hbar^2 |\psi|^2 R m^{-1} \ln(R/\xi)$, see [6, 12], and the momentum is $p^{\text{vort}} = 2\pi^2 \hbar |\psi|^2 R^2$, m here is the mass of the pair for systems with pairing, and the mass of the boson quasiparticle in bosonic superfluids, *e.g.*, the mass of the ^4He atom in case of the He-II, R is the radius of the vortex ring, and $\xi \sim c_0^{1/2} a_0^{-1/2} t^{-\alpha/2}$ is the minimal length scale associated with the mother condensate. Thus, $v_{c1} = \epsilon^{\text{vort}} / p^{\text{vort}} = \hbar (Rm)^{-1} \ln(R/\xi)$, is the Landau critical velocity for the vortex production, where now R is the distance of the order of the size of the system. For $v > v_{c1}$ the vortex rings are pushed out of the medium, if the density profile has even slight inhomogeneity. Note that for spatially extended systems the value v_{c1} is usually lower than the Landau critical velocity v_c^L . The flow moving with the velocity v for $v_{c1} \leq v \leq v_{c2}$ may be considered as metastable, since the vortex creation probability is hindered by a large potential barrier and formation of a vortex takes a long time [13]. Note that already for v just slightly exceeding v_{c1} , the number of the produced vortices may become sufficiently large and their interaction forces the normal and superfluid components to move as a solid, with the same velocity, even if initially they have had different velocities. In the exterior regions of the vortex cores the superfluidity still persists and our consideration of the condensation of excitations in the velocity interval $v_c^L < v < v_{c2}$ is applicable.

In the presence of the condensate of excitations the density becomes spatially oscillating around its averaged value. For a weak condensate, *i.e.*, $|v - v_c^L| \ll v_c^L$, we find perturbatively $\delta n = n - \bar{n} \approx 2\sqrt{n} |\psi'_0| \cos((\epsilon(k_0)t - \vec{k}_0 \vec{r})/\hbar)$. Such a density modulation predicted in [1] was reproduced in the numerical simulation of the supercritical flow in He-II using the realistic density functional [14].

The above consideration holds for any superfluid with the conserved number of particles in the mother condensate plus the condensate of excitations, *e.g.*, for the cold Bose gases, if the spectrum of the over-condensate excitations is such that $\epsilon(k_0)/k_0$ has a minimum at $k = k_0 \neq 0$, as has been conjectured in [4].

In case of the Fermi systems with pairing, for the bosonic modes (Anderson-Bogoliubov, Schmid and Carlson-Goldman ones) with the excitation energy $\simeq 2\Delta$, cf. [10], the maximum momentum, up to which the mode is not yet destroyed, is $k_0 \simeq 2p_F$, cf. [9, 15]. Hence, for these modes the Landau critical velocity is $v_c^L \simeq \Delta/p_F$, and for $v > v_c^L$ there is a chance for the condensation of the Bose excitations as we considered above. Besides bosonic excitations there exist fermionic ones with the spectrum $\epsilon_f(p) = \sqrt{\Delta^2 + v_F^2(p - p_F)^2}$. Stemming from the breakup of Cooper pairs, the fermionic excitations are produced pairwise. Therefore, the corresponding (fermion) Landau critical velocity is $v_{c,f}^L = \min_{\vec{p}_1, \vec{p}_2} [(\epsilon_f(p_1) + \epsilon_f(p_2))/|\vec{p}_1 + \vec{p}_2|]$. The latter expression reduces to [16] $v_{c,f}^L = (\Delta/p_F)/(1 + \Delta^2/p_F^2 v_F^2)^{1/2}$. We

see that up to a small correction $v_{c,f}^L \simeq v_c^L$. For $T \rightarrow 0$ and $0 \leq v/v_c^L - 1 \ll 1$ we find $\langle \vec{p} \vec{v} \rangle / \rho \simeq 2\sqrt{2v_c^L}(v - v_c^L)^{3/2}$ and the energy gain due to the fermion pair breaking is

$$\begin{aligned} \delta \bar{E}_{\text{pair}} &= \int \frac{2d^3p}{(2\pi\hbar)^3} (\epsilon_f(p) - \vec{p} \vec{v}) \theta(\epsilon_f(p) - \vec{p} \vec{v}) \\ &\approx -2\sqrt{2}\rho(v - v_c^L)^2 [v/v_c^L - 1]^{1/2}. \end{aligned} \quad (10)$$

Moreover, for $v > v_{c,f}^L$ the pairing gap decreases with increase of v as [17] $\Delta(v)/\Delta \approx 1 - (3/2)(v/v_c^L - 1)^2$, reaching zero for $v = v_{c2,f}^L = \frac{6}{5}v_c^L$ (Rogers-Bardeen effect [18]). The energy gain (10) is less than (9) and the production of the condensate of Bose excitations is energetically more profitable than the Cooper pair breaking. Since in the presence of the condensate of excitations $v_{\text{fin}} = v_c^L$ an additional energy gain due to the appearance of the latter is: $F_{\text{GL}}^{\text{eq}}(T = 0, \Delta) - F_{\text{GL}}^{\text{eq}}(T = 0, \Delta(v)) \approx -(9/8)\rho(v - v_c^L)^2$, for $0 \leq v/v_c^L - 1 \ll 1$. For $v > v_{c2,f}^L$ the gain becomes $F_{\text{GL}}^{\text{eq}}(T = 0, \Delta) = -3\rho(v_c^L)^2/4$.

Moving superfluid-normal composites. – We turn now to the case of systems consisting of normal and superfluid parts, like He-II at finite temperature, metallic superconductors, or neutron star matter. Here, the number of particles in the condensate is not conserved even at $v = 0$ since a part of particles can be transferred from the superfluid to the normal subsystem. The state of the superfluid subsystem is described by the GL functional (1). We assume that *the normal and superfluid subsystems move with the same velocity \vec{v}* with respect to a laboratory frame. This means that we consider the motion of the fluid during the time τ shorter than the typical friction time, $\tau_{\text{fr}}^{\text{norm}}$, at which the normal component is decelerated, if the fluid has a contact with the wall.

Minimization over the order parameter in the rest frame of the fluid yields

$$|\psi_{v=0}^{\text{eq}}|^2 = a(t)/b(t), \quad F_{\text{GL}}^{\text{eq}}[\psi_{v=0}^{\text{eq}}] = -b(t)|\psi_{v=0}^{\text{eq}}|^4/2. \quad (11)$$

In the absence of the population of excitation modes the superfluid and normal subsystems decouple. In this case the initial free-energy density of the system is given by

$$F_{\text{in}} = \rho v^2/2 + F_{\text{bind}} - a^2(t)/(2b(t)). \quad (12)$$

Here ρ is the total (normal+superfluid) mass density, $\rho = \rho_n(v=0) + m|\psi(v=0)|^2$, and F_{bind} is a binding free-energy density of the normal subsystem in its rest reference frame (coinciding with the rest frame of the superfluid in the case under consideration). The explicit form of F_{bind} is not of our interest.

When the condensate of excitations is formed, the initial momentum density is redistributed between the fluid and the condensate of excitations:

$$\rho \vec{v} = (\rho - m|\psi'_0|^2) \vec{v}_{\text{fin}} + (\vec{k}_0 + m\vec{v}_{\text{fin}}) |\psi'_0|^2. \quad (13)$$

Here we assume (as argued below) that after the appearance of the condensate of excitations in the form (3) the

normal and superfluid subsystems continue to move with one and the same velocity \vec{v}_{fin} . In the presence of the condensate of excitations the free energy density becomes

$$\begin{aligned} F_{\text{fin}} &= \frac{1}{2}\rho v_{\text{fin}}^2 + F_{\text{bind}} + F_{\text{GL}}[\psi, \nabla\psi = 0] \\ &\quad + (\epsilon(k_0) + \epsilon_{\text{bind}}) |\psi'_0|^2 + 2b(t) |\psi|^2 |\psi'_0|^2 + \frac{1}{2}b(t) |\psi'_0|^4. \end{aligned} \quad (14)$$

To get this expression we used eq. (1) and replaced there ψ with $\psi_{\text{fin}} = \psi + \psi'$. The energy of excitations should be counted here from the excitation energy on top of the mother condensate at rest determined by eq. (11), $\epsilon_{\text{bind}} = \partial F_{\text{GL}}^{\text{eq}}[\psi = \psi_{v=0}^{\text{eq}} + \psi'] / \partial |\psi'|^2|_{\psi'=0} = -2a(t)$.

Now, using the momentum conservation (13) we express \vec{v}_{fin} through \vec{v} and get for the change of the volume-averaged free-energy density associated with the appearance of the condensate of excitations,

$$\begin{aligned} \delta \bar{F} &= \frac{1}{2}b(t) (|\psi|^2 - a(t)/b(t))^2 + (\epsilon(k_0) - k_0 v) |\psi'_0|^2 \\ &\quad + 2b(t) (|\psi|^2 - a(t)/b(t)) |\psi'_0|^2 + \frac{1}{2}\tilde{b}(t) |\psi'_0|^4, \end{aligned} \quad (15)$$

where $\tilde{b}(t) = b(t) + k_0^2/\rho$ and we put $\vec{k}_0 \parallel \vec{v}$. We note that the normal subsystem serves as a reservoir of particles at the formation of the condensates, which amplitudes are chosen by minimization of the free energy of the system. Therefore, we vary $\delta \bar{F}$ with respect to ψ and ψ'_0 independently. Thus, minimizing (15) we find

$$\begin{aligned} |\psi'_0|^2 &= \frac{k_0(v - v_c^L)}{k_0^2/\rho - 3b(t)} \theta(v - v_c^L) \theta(k_0^2/\rho - 3b(t)), \quad (16) \\ |\psi|^2 &= (a(t)/b(t) - 2|\psi'_0|^2) \theta(\tilde{T}_c(v) - T) \theta(v_{c2}(t) - v). \end{aligned}$$

The quantity \tilde{T}_c stands for the renormalized critical temperature, which depends now on the flow velocity, and $v_{c2}(t)$ stands for the second critical velocity depending on T . The condition $|\psi|^2 = 0$ implies the relation between v and T

$$v = v_c^L + a(t)k_0/(2b(t)\rho) - 3a(t)/(2k_0). \quad (17)$$

The solution of this equation for the velocity, $v_{c2}(t)$, increases with the decreasing temperature, and the solution for the temperature, $\tilde{T}_c(v)$, decreases with increasing v . At $T = \tilde{T}_c(v)$ or $v = v_{c2}(t)$ we have $|\psi|^2 = 0$ but $|\psi'_0|^2 \neq 0$, and for $T > \tilde{T}_c(v)$ or for $v > v_{c2}(t)$ the condensate $|\psi'_0|^2$ vanishes, if, as we assume, for $|\psi|^2 = 0$ the spectrum of excitations does not contain a low-lying branch. Thus, the superfluidity is destroyed at $T = \tilde{T}_c(v)$ or $v = v_{c2}(t)$ in a first-order phase transition.

From (13) and (16) we find for $v > v_c^L$ and $k_0^2/(\rho b(t)) > 3$ the resulting velocity of the flow

$$v_{\text{fin}} = v_c^L - (v - v_c^L)/(k_0^2/(3b(t)\rho) - 1) < v_c^L. \quad (18)$$

Substituting the order parameters from (16) in (15), we find for the volume-averaged free-energy density gain owing to appearance of the condensate of excitations

$$\delta \bar{F} = -\frac{\rho}{2} \frac{(v - v_c^L)^2}{1 - 3b(t)\rho/k_0^2} \theta(v - v_c^L) \theta(v_{c2} - v) \quad (19)$$

for $k_0^2/\rho > 3b(t)$. Thus, for $v_c^L < v < v_{c2}$ the free energy decreases owing to the appearance of the condensate of excitations with $k \neq 0$ in the presence of the non-vanishing mother condensate. The value of k_0 is to be found from the minimization of eq. (19). As \tilde{T}_c , the condensate momentum k_0 gets renormalized and differs now from the value corresponding to the minimum of $\epsilon(k)/k$. For $k_0^2/\rho \gg 3b_0$ the expression (19) for the gain in the volume-averaged free-energy density transforms at $T = 0$ into the expression (9) for the gain in the volume-averaged energy density. As in the case of $T = 0$, for $T \neq 0$ the condensate of excitations appears at $v = v_c^L$ in the second-order phase transition but it disappears at $v = v_{c2}$ in the first-order phase transition with the jumps

$$\delta\bar{F}(v_{c2}) \approx \frac{a^2(t)k_0^2}{8b^2(t)\rho}, \quad |\psi'_0(v_{c2})|^2 = \frac{a(t)}{2b(t)}. \quad (20)$$

The dynamics of the condensate of excitations is determined by the equation $\Gamma\psi' = -\delta\bar{F}/\delta\psi^*$. We emphasize that the above consideration assumes that the formation rate Γ of the condensate of excitations is faster than the deceleration rate $1/\tau_{\text{fr}}^{\text{norm}}$ of the normal subsystem.

When a homogeneous fluid flowing with $v > v_c^L$ at $T > \tilde{T}_c(v)$ is cooled down to $T < \tilde{T}_c(v)$, it consists of three components: the normal and superfluid ones and the condensate of excitations, all moving with $v_{\text{fin}} < v_c^L$. If the system is then re-heated to $T > \tilde{T}_c(v)$, the superfluidity and the condensate of excitations vanish and the remaining normal fluid consists of two fractions: one is moving with $v_{\text{fin}}(\tilde{T}_c) < v_c^L$ and the other one, $\delta n = a(\tilde{T}_c)/(2b(\tilde{T}_c))$, is moving with a higher velocity until a new equilibrium is established. Note also that for fermion superfluids at $T \neq 0$ after the condensate of excitations is formed the flow velocity $v_{\text{fin}} < v_{c,f}^L$, for $v - v_c^L > 4tv_c^L/9$, and thereby the Cooper pair breaking does not occur, whereas the condensate of Bose excitations is preserved.

Estimates for fermionic superfluids and He-II. –

We apply now expressions derived in the previous section to several practical cases. First, we consider a fermion system with pairing. With the BCS parameters (2) we estimate $b_0\rho/k_0^2 = 3\Delta^2/(8v_F^2p_F^2)$ and $a_0/k_0 = 3\Delta^2/(4v_Fp_F^2)$, where $\rho \simeq nm_F$. We see that inequality $k_0^2/\rho \gg 3b_0$ is reduced to inequality $\Delta \ll \epsilon_F$, which is well satisfied. In this limit $|\psi_0|^2$ given by eq. (16) gets the same form as eq. (8). The resulting flow velocity after condensation, (18), is lower than the Landau critical velocity but close to it, $v_{\text{fin}} \simeq v_c^L - 9(v_c^L)^2(v - v_c^L)/(8v_F^2)$.

Since for the BCS case we have $\alpha = 1, \beta = 0$, eq. (17) for the new critical temperature is easily solved, for $v > v_c^L$

$$\frac{\tilde{T}_c}{T_c} = 1 - \frac{2k_0b_0(v - v_c^L)}{a_0(k_0^2/\rho - 3b_0)} \approx 1 - \frac{v - v_c^L}{v_F}. \quad (21)$$

We also estimate the maximal second critical velocity as $v_{c2}^{\text{max}} \simeq v_c^L + v_F$.

We turn now to the bosonic superfluid – helium-II. Making use of the values of the GL parameters presented above

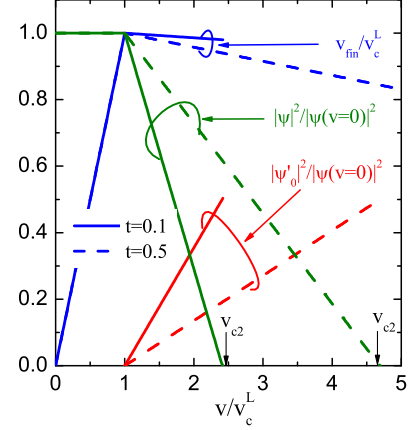


Fig. 1: Condensate amplitudes $|\psi|^2$ and $|\psi'_0|^2$, eq. (16), and the final flow velocity v_{fin} , eq. (18), in superfluid ^4He plotted as functions of the flow velocity for various temperatures, $t = (T_c - T)/T_c$. Vertical arrows indicate the values of the second critical velocity v_{c2} . Velocities are scaled by the values of the Landau critical velocities $v_c^L(t = 0.5) = 59 \text{ m/s}$ and $v_c^L(t = 0.1) = 55 \text{ m/s}$, and the condensates are normalized to the condensate amplitude in the superfluid at rest, eq. (11).

and taking into account that we deal with the rotonic excitation, i.e., $k_0 \simeq k_r$ and $\epsilon(k_0) \simeq \Delta_r$, we estimate, $k_0^2/(b_0\rho) \simeq 47$, $v_c^L(T \rightarrow 0) \simeq 60 \text{ m/s}$, $a_0/k_0 \simeq 16 \text{ m/s}$. Using the results of [7] the temperature dependence of v_c^L can be fitted with 99% accuracy as $v_c^L(T)/v_c^L(0) \simeq 1 - 0.7e^{-2.14/\tilde{t}} + 200\tilde{t}e^{-8/\tilde{t}}$, where $\tilde{t} = T/T_c$. Using these parameters we evaluate the condensate amplitudes and the final flow velocity as functions of temperature and depict them in Fig. 1. The condensate of excitations appears at $v = v_c^L$ in a second-order phase transition. For $v > v_c^L$ the amplitude of the condensate $|\psi'_0|^2$ ($|\psi|^2$) increases (decreases) linearly with v . The closer T is to T_c , the steeper the change of the condensate amplitudes is. The final velocity of the flow, which sets in after the appearance of the condensate of excitations, decreases with the increase of v . For He-II, we have $\alpha = 4/3, \beta = 2/3$ and the renormalized critical temperature determined by eq. (17) is for $v > v_c^L$:

$$\frac{\tilde{T}_c}{T_c} = 1 - \left[\frac{k_0^2}{6b_0\rho} - \sqrt{\frac{k_0^4}{36b_0^2\rho^2} - \frac{2k_0}{3a_0}[v - v_c^L]} \right]^{3/2} \approx 1 - 0.05(v/v_c^L(T_c) - 1)^{3/2}. \quad (22)$$

The mother condensate $|\psi|^2$ vanishes when v reaches the value of the second critical velocity v_{c2} , which depends on the temperature as $v_{c2} \approx v_c^L(t) + (363t^{2/3} - 23.5t^{4/3})\text{m/s}$. At $v = v_{c2}$ the superfluidity disappears in a first-order phase transition. The corresponding energy release can be estimated from (20) as $\delta F(v_{c2}) \approx \frac{47a_0^2}{8b_0}t^{4/3} \simeq 5.9t^{4/3}(T_c\Delta C_p)$, where $\Delta C_p = 0.76 \cdot 10^7 \text{ erg}/(\text{cm}^3\text{K})$ is the specific heat jump at T_c [6].

Rotating superfluids. Pulsars. – The novel phase with the condensate of excitations may also exist in rotat-

ing systems. Excitations can be generated because of the rotation. Presence of friction with an external wall or difference between velocities of the superfluid and the normal fluid are not necessarily required to produce excitations. Now we should use the angular momentum conservation instead of the momentum conservation. The structure of the order parameter is more complicated than the plane wave [2]. With these modifications, the results, which we obtained above for the motion with the constant \vec{v} , continue to hold. The value of the critical angular velocity $\Omega_{c1} \sim v_{c1}/R$ is very low for systems of a large size R .

In the inner crust and in a part of the core of a neutron star, protons and neutrons are paired in the $1S_0$ state owing to attractive pp and nn interactions. In denser regions of the star interior the $1S_0$ pairing disappears but neutrons might be paired in the $3P_2$ state [19]. The charged superfluid should co-rotate with the normal matter without forming vortices, this results in the appearance of a tiny magnetic field $\vec{h} = 2m_p\vec{\Omega}/e_p$ (London effect) in the whole volume of the superfluid, m_p (e_p) is the proton mass (charge) [19]. This tiny field, being $\lesssim 10^{-2}G$ for the most rapidly rotating pulsars, has no influence on the relevant physical quantities and can be neglected.

With the typical neutron star radius, $R \sim 10$ km, and for $\Delta \sim \text{MeV}$ typical for the $1S_0$ nn pairing, we estimate $\Omega_{c1} \sim 10^{-14}$ Hz. For $\Omega > \Omega_{c1}$ the neutron star contains arrays of neutron vortices with regions of the superfluidity among them, and the star as a whole rotates as a rigid body. The vortices would cover the whole space, only if Ω reaches unrealistically large value $\Omega_{c2}^{\text{vort}} \sim 10^{20}$ Hz. The most rapidly rotating pulsar PSR J1748-2446ad has the angular velocity 4500 Hz [20]. The value of the critical angular velocity for the formation of the condensate of excitations in the neutron star matter is $\Omega_c \sim \Omega_c^L \simeq \Delta/(p_F R) \sim 10^2$ Hz for the pairing gap $\Delta \sim \text{MeV}$ and $p_F \sim 300 \text{ MeV}/c$ at the nucleon density $n \sim n_0$, where n_0 is the density of the atomic nucleus, and c is the speed of light. The superfluidity continues to coexist with the condensate of excitations and the array of vortices until the rotation frequency Ω reaches the value $\Omega_{c2} > \Omega_c^L$, at which both the condensate of excitations and the superfluidity disappear completely. From eq. (17) with the BCS parameters we estimate $\Omega_{c2} \sim v_{c2}/R \lesssim 10^4$ Hz. Thus, in the detected rapidly rotating pulsars the condensate of excitations might coexist with superfluidity.

Conclusion. — In this letter we studied a possibility of the condensation of excitations with $k \neq 0$, when a superfluid flows with a velocity larger than the Landau critical velocity, $v > v_c^L$. We included an interaction between the “mother” condensate of the superfluid and the condensate of excitations and considered the superfluid at zero and finite temperatures. We assumed that the superfluid and normal components move with equal velocities. In practice it might be achieved for $v > v_{c1}$, where v_{c1} is the critical velocity of the vortex formation. We found that the condensate of excitations appears in a second-order

phase transition at $v = v_c^L$ and the condensate amplitude grows linearly with the increasing velocity. Simultaneously the mother condensate decreases and vanishes at $v = v_{c2}$, then the superfluidity is destroyed in a first-order phase transition with an energy release. For $v_c^L < v < v_{c2}$ the resulting flow velocity is $v_{\text{fin}} \leq v_c^L$, whereby the equality is realized for $T = 0$. We argued that for the fermion superfluids the condensate of bosonic excitations might be stable against the appearance of fermionic excitations from the Cooper-pair breaking. Finally, we considered condensation of excitations in rotating superfluid systems, such as pulsars. Our estimates show that superfluidity might coexist with the condensate of excitations in the rapidly rotating pulsars.

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